STA 623 — Fall 2011 — Dr. Charnigo

Written Assignment 1 Solutions

1. Suppose $x \in A \cup (B \cap C)$. Then either $x \in A$ or $x \in (B \cap C)$. In the former case, $x \in (A \cup B)$ and $x \in (A \cup C)$. In the latter case, $x \in B$ and $x \in C$, so that, again, $x \in (A \cup B)$ and $x \in (A \cup C)$. Thus, $x \in (A \cup B) \cap (A \cup C)$. Thus, $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Suppose $x \in (A \cup B) \cap (A \cup C)$. Then $x \in (A \cup B)$ and $x \in (A \cup C)$. If $x \in A$, then $x \in A \cup (B \cap C)$ and we are done. Otherwise, we must have $x \in B$ and $x \in C$, whence $x \in (B \cap C)$ and $x \in A \cup (B \cap C)$. Thus, $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

2. By result 7, we have $P(A \cap B) + P(A^c \cap B) = P(B)$. By axiom 1, we have $P(A^c \cap B) \geq 0$, so that $P(A \cap B) \leq P(B)$. Now, if $A \subset B$, then $A \cap B = A$. Thus, if $A \subset B$, then $P(A) \leq P(B)$.

3. The number of possible hands is $\binom{52}{7} = 133784560$. The number of possible hands with one triple and two pairs is 123552, computed as follows: $\binom{13}{1} = 13$ ways to choose denominations for the triple, $\binom{12}{2} = 66$ ways to choose denominations for the two pairs, $\binom{4}{1} = 4$ ways to choose suits for the triple, and $\binom{4}{2}^2 = 36$ ways to choose suits for the two pairs. Thus, the requested probability is $123552/133784560 \approx 9.24 \times 10^{-4}$.

4. The probability of winning the first game is $[\binom{27}{0}\binom{3}{3} + \binom{27}{1}\binom{3}{2}]/\binom{30}{3} = 82/4060 \approx 0.020$, calculated by noting that there are: $\binom{30}{3}$ ways for the player to select 3 numbers; $\binom{3}{2}$ ways for the player to select 2 of the drawn numbers and $\binom{27}{1}$ ways for the player to select 1 of the non-drawn numbers [in which case, the player wins by matching 2]; and, $\binom{3}{3}$ ways for the player to select 3 of the drawn numbers and $\binom{27}{0}$ ways for the player to select 0 of the non-drawn numbers [in which case, the player wins by matching 3]. This is much greater than the probability of winning the second game, $[\binom{54}{0}\binom{6}{6}+\binom{54}{1}\binom{6}{5}+\binom{54}{2}\binom{6}{4}]/\binom{60}{6} = 21790/50063860 \approx 4.35\times10^{-4}$, calculated similarly.

5. Here is one possible answer. Let $S := \{1, 2, 3, 4, 5, 6, 7, 8\}$ with $P(s) := 1/8$ for each $s \in S$ and $P(D) := \sum_{s \in D} P(s)$ for any event $D \subset S$. Let $A := \{1, 2, 3, 4\}$, $B := \{2, 4, 6, 8\}$, and $C := \{2, 4, 5, 7\}$. Then $A \cap B = A \cap C = B \cap C = A \cap B \cap C = \{2, 4\}$ so that $1/4 = P(A \cap B) = P(A)P(B) = 1/2 \times 1/2$ [A and B independent], $1/4 = P(A \cap C) = P(A)P(C) = 1/2 \times 1/2$ [A and C independent], and $1/4 = P(B \cap C) = P(B)P(C) = 1/2 \times 1/2$ [B and C independent], but $1/4 = P(A \cap B \cap C) \neq P(A)P(B)P(C) = 1/2 \times 1/2 \times 1/2$ [A, B, and C not independent].

6. If the null hypothesis in post-hoc test i is true, then $P(A_i \cap B_i) = P(A_i|B_i)P(B_i) \leq \alpha/m \times 1 = \alpha/m$. If the null hypothesis in post-hoc test i is false, then $P(A_i \cap B_i) \leq P(B_i) = 0 \leq \alpha/m$. Thus, by applying result 10, we find that $P(\cup_{i=1}^m \{A_i \cap B_i\}) \leq \sum_{i=1}^m P(A_i \cap B_i) \leq \sum_{i=1}^m \alpha/m = \alpha$.

7. A false positive occurs when someone who really does not have HIV tests positive, which can occur if $Q < 100$ but not if $Q = 100$. A false negative occurs when someone who really does have HIV tests negative, which can occur if $P < 100$ but not if $P = 100$. Thus, only statements b and c are true.

8. All definitions clearly have $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1$ regardless of C, so to determine whether each definition yields a valid cumulative distribution function (cdf) we only need to check whether $F(x)$ is nondecreasing and right continuous. Moreover, because trigonometric functions are continuous, the only possible violations of right continuity are at 0 and 1.

a. If $C < 0$ or if $C > \pi/2$, then $F(x)$ is decreasing on some subinterval of $(0, 1)$ so that $F(x)$ cannot be a valid cdf. If $0 \leq C < \pi/2$, then $F(x)$ is not right continuous at 1 so that $F(x)$ cannot be a valid cdf. If $C = \pi/2$, then $F(x)$ is nondecreasing and continuous, and therefore $F(x)$ is a valid cdf for a continuous random variable.

b. If $C \neq 0$, then $F(x)$ is decreasing on some subinterval of $(0, 1)$ so that $F(x)$ cannot be a valid cdf. If $C = 0$, then $F(x)$ simplifies to $1_{x\geq 0}$, a right continuous and nondecreasing step function, and therefore $F(x)$ is a valid cdf for a discrete random variable.

c. If $C < 0$ or if $C > \pi/2$, then $F(x)$ is decreasing on some subinterval of $(0, 1)$ so that $F(x)$ cannot be a valid cdf. If $0 < C < \pi/2$, then $F(x)$ is nondecreasing and right continuous, but $F(x)$ is neither a step function nor continuous at 1, and therefore $F(x)$ is a valid cdf for a random variable that is neither discrete nor continuous. If $C = \pi/2$, then $F(x)$ is nondecreasing and continuous, and therefore $F(x)$ is a valid cdf for a continuous random variable. If $C = 0$, then $F(x)$ simplifies to $1_{x\geq 1}$, a right continuous and nondecreasing step function, and therefore $F(x)$ is a valid cdf for a discrete random variable.

d. If $C \neq 0$, then $F(x)$ is decreasing on some subinterval of $(0, 1)$ so that $F(x)$ cannot be a valid cdf. If $C = 0$, then $F(x)$ simplifies to $1_{x>0}$, which is not right continuous, and therefore $F(x)$ cannot be a valid cdf.