STA 623 — Fall 2011 — Dr. Charnigo

Written Assignment 2 Solutions

Put $f_X(x) := C(\log x)^{-\alpha}x^{-1}1_{\{x \geq e\}}$. Note that $f_X(x)$ cannot be a probability density function if $C < 0$ (because then $f_X(x) < 0$ for $x \geq e$) or if $C = 0$ (because then $f_X(x)$ is identically 0 and thus cannot integrate to 1), so hereafter we assume that $C > 0$.

1. We have $\int_e^\infty C(\log x)^{-\alpha}x^{-1}\ dx = \lim_{M\to\infty}\int_e^M C(\log x)^{-\alpha}x^{-1}\ dx$. Making the substitution $u := \log x$ and $du := x^{-1}\ dx$, we have $\lim_{M\to\infty}\int_1^{\log M} Cu^{-\alpha}\ du$. If $\alpha = 1$, then this is $\lim_{M\to\infty}\int_1^{\log M} Cu^{-1}\,du = \lim_{M\to\infty} C\log(\log M) = \infty$. Otherwise, this is $\lim_{M\to\infty}\int_1^{\log M} Cu^{-\alpha}\,du = \lim_{M\to\infty} C\{(\log M)^{1-\alpha} - 1\}/(1-\alpha)$. If $\alpha < 1$, then this limit is ∞. If $\alpha > 1$, then this limit is $-C/(1-\alpha) = C/(\alpha - 1)$. As such, we may obtain a probability density function when $\alpha > 1$ by taking $C := \alpha - 1$, since this choice of C yields $\int_e^\infty f_X(x)\ dx = 1$.

2. Put $g(x) := (\log x)^3$ and $h(x) := x$. We may apply L'Hopital's rule to conclude that $\lim_{x\to\infty} g(x)/h(x) = \lim_{x\to\infty} g'(x)/h'(x)$, assuming that the latter limit exists (which will be demonstrated by subsequent computations). We have $g'(x) = 3(\log x)^2x^{-1}$ and $h'(x) = 1$, so that $\lim_{x\to\infty} g'(x)/h'(x) = \lim_{x\to\infty} 3(\log x)^2x^{-1}$. Two more invocations of L'Hopital's rule yield $\lim_{x\to\infty} 3(\log x)^2x^{-1} = \lim_{x\to\infty} 6(\log x)x^{-1} = \lim_{x\to\infty} 6x^{-1}$. The last limit is clearly 0, so that all of the preceding limits must also exist and equal 0.

By the definition of limit, there exists $L \in (0,\infty)$ such that $|(\log x)^3x^{-1} - 0| \leq 1$ whenever $x \geq L$. Put $K := \max\{L, e\}$. Since $|(\log x)^3x^{-1}| = (\log x)^3x^{-1}$ whenever $x \geq e$, we have $(\log x)^3x^{-1} \leq 1$ whenever $x \geq K$. As such, $x^{-1} \leq (\log x)^{-3}$ whenever $x \geq K$.

3. Since $x^{-1} \leq (\log x)^{-3}$ whenever $x \geq K$, we have $\int_K^\infty x^{-1}\ dx \leq \int_K^\infty (\log x)^{-3}\ dx$. Since the former integral is infinite, so is the latter integral. Moreover, since $0 \leq (\log x)^{-3} \leq 1$ when $x \geq e$, we know that $\int_e^K (\log x)^{-3}\ dx$ is finite (and, more specifically, between 0 and $K - e$). Thus, $E[X] = \int_e^\infty Cx(\log x)^{-3}x^{-1}\ dx = C\int_e^K (\log x)^{-3}\ dx + C\int_K^\infty (\log x)^{-3}\ dx = \infty$, since adding a finite quantity to an infinite quantity yields an infinite quantity.

4. We have $E[\log X] = \int_e^\infty C(\log x)(\log x)^{-3}x^{-1}\ dx = \int_e^\infty C(\log x)^{-2}x^{-1}\ dx$. Making the substitution $u := \log x$ and $du := x^{-1}\ dx$, we have $\int_1^\infty Cu^{-2}\ du = C = 2$.

5. Put $g(x) := \log x$ for $x \in \mathcal{X} := [e, \infty)$. Then g is a strictly increasing function with $g^{-1}(y) = e^y$ for $y \in \mathcal{Y} := [1,\infty)$. So, $f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) = C(\log e^y)^{-3}(e^y)^{-1}\frac{d}{dy}e^y = Cy^{-3} = 2y^{-3}$ for $y \in \mathcal{Y}$.

6. We have $E[Y] = \int_1^\infty y f_Y(y)\ dy = \int_1^\infty y\ 2y^{-3}\ dy = 2\int_1^\infty y^{-2}\ dy = 2$. The answer is the same as in part d because $\log X$ and Y are the same random variable and, as such, must have the same expected value.

7. We have $E[Y^2] = \int_1^\infty y^2 f_Y(y)\ dy = \int_1^\infty y^2\ 2y^{-3}\ dy = 2\int_1^\infty y^{-1}\ dy = \infty$.

8. The cumulative distribution function of Y is $F_Y(y) := \int_1^y f_Y(t)\ dt = \int_1^y 2t^{-3}\ dt = 1 - y^{-2}$ for $y \in \mathcal{Y}$.

9. If $Z := \lfloor Y \rfloor$, then the probability mass function of Z is $f_Z(z) = P(\lfloor Y \rfloor = z) = P(z \leq Y < z+1) = F_Y(z+1) - F_Y(z) = z^{-2} - (z+1)^{-2}$ for $z \in \{1, 2, 3, \ldots\}$.

10. We have $0 \leq Z \leq Y$. Since Y has finite expected value per exercise 6 and since 0 obviously has finite expected value, we conclude that Z also has finite expected value. More specifically, we have $0 = E[0] \leq E[Z] \leq E[Y] = 2$.