STA 623 — Fall 2011 — Dr. Charnigo

Written Assignment 3

Written Assignment 3 is due on Tuesday 01 November at the end of class. You are encouraged to work in groups of two or three, but you may work individually if you prefer.

[50] 1. Let X_n have the chi-square distribution on n degrees of freedom for $n \in \{1, 2, \ldots\}$. In what follows, you may quote without proof the facts that X_n has mean n, variance $2n$, and moment generating function $M_{X_n}(t) := (1 - 2t)^{-n/2}$ for $t < 1/2$.

[10] a. Put $Z_n := (2n)^{-1/2}(X_n - n)$. What is $M_{Z_n}(t)$?

[10] b. What is $\lim_{n\to\infty} M_{Z_n}(t)$?

[10] c. Use result b to ascertain $\lim_{n\to\infty} P(X_n \leq z(2n)^{1/2} + n)$ for any fixed real number z. (You may express your answer in integral form.)

[10] d. What does result c tell you about the shape of the chi-square distribution on n degrees of freedom when n is large?

[10] e. Employ conclusion d to obtain an approximation to $P(180 \leq X_{200} \leq 220)$. Then use software, such as R or SAS, to calculate $P(180 \leq X_{200} \leq 220)$ to four decimal places. Repeat the preceding steps for $P(0 \leq X_2 \leq 4)$. Thereby observe that conclusion d is not applicable when n is small.

[50] 2. Consider the same scenario as in exercise 1, except that now n can be any positive real number.

[10] a. Show that the distribution of X_n belongs to an exponential family.

[10] b. Use fact a to calculate $E[\log X_n]$. (You may express your answer in terms of the digamma function, $\psi[n] := \frac{d}{dt} \log \Gamma[t]|_{t=n}$.)

[10] c. Use fact a to calculate $Var[\log X_n]$. (You may express your answer in terms of the trigamma function, $\psi_1[n] := \frac{d^2}{dt^2} \log \Gamma[t]|_{t=n}$.)

[10] d. Apply result b to evaluate the (improper) definite integral $\int_0^\infty \log[x]x^{n/2-1} \exp[-x/2]\, dx$.

[10] e. Use result d to obtain $\int_0^\infty \log[x]x^{\alpha-1} \exp[-\lambda x]\, dx$, where α and λ can be any positive real numbers. Thereby observe that some definite integrals not amenable to evaluation by the Fundamental Theorem of Calculus are nonetheless tractable upon invocation of probability theory.