STA 623 — Fall 2011 — Dr. Charnigo

Written Assignment 3 Solutions

1a. We have

$$\begin{aligned} M_{Z_n}(t) &= E[\exp(tZ_n)] \\ &= E[\exp(t(2n)^{-1/2}X_n)]\exp(-tn(2n)^{-1/2}) \\ &= M_{X_n}(t(2n)^{-1/2})\exp(-t\sqrt{n/2}) \\ &= (1-t\sqrt{2/n})^{-n/2}\exp(-t\sqrt{n/2}) \end{aligned}$$

for $t < \sqrt{n/2}$.

1b. In what follows, suppose that $|t| \in (0, 1/2]$. Note that $\exp(-t\sqrt{n/2}) = \exp(t\sqrt{2/n})^{-n/2}$ and

$$\begin{aligned} (1-t\sqrt{2/n})\exp(t\sqrt{2/n}) &= (1-t\sqrt{2/n})/\exp(-t\sqrt{2/n}) \\ &= 1-(\exp(-t\sqrt{2/n})-1+t\sqrt{2/n})/\exp(-t\sqrt{2/n}) \\ &= 1-(\{1+o(1)\}t^2/n)/\{1+o(1)\} \\ &= 1-(\{1+o(1)\}t^2/n), \end{aligned}$$

where $o(1) \to 0$ as $t\sqrt{2/n} \to 0$. Above, we have used the facts that $\exp(-u) = 1-u+\{1+o(1)\}u^2/2$ and $\exp(-u) = \{1+o(1)\}$ as $u \to 0$. Thus, we have

$$\begin{aligned} M_{Z_n}(t) &= [(1-t\sqrt{2/n})\exp(t\sqrt{2/n})]^{-n/2} \\ &= [1-(\{1+o(1)\}t^2/n)]^{-n/2} \\ &= ([1-(\{1+o(1)\}t^2/n)]^n)^{-1/2} \\ &\to \exp[-t^2]^{-1/2} \\ &= \exp[t^2/2]. \end{aligned}$$

Above, we have used the fact that $(1+a\{1+o(1)\}/n)^n \to \exp[a]$ as $n \to \infty$.

1c. Since $M_{Z_n}(t) \to M_Z(t)$ for all $t \in [-1/2, 1/2]$, where Z has the standard normal distribution, we have

$$P(X_n \le z(2n)^{1/2}+n) = P(Z_n \le z) \to P(Z \le z) = \int_{-\infty}^{z} (2\pi)^{-1/2}\exp[-t^2/2]\,dt.$$

1d. The chi-square distribution on n degrees of freedom is approximately bell-shaped when n is large.

1e. The approximate probabilities are

$$\begin{aligned} P(180 \le X_{200} \le 220) &= P((180-200)/20 \le (X_{200}-200)/20 \le (220-200)/20) \\ &= P(-1 \le (X_{200}-200)/20 \le 1) \\ &\approx P(-1 \le Z \le 1) = 0.6827. \end{aligned}$$

Similarly, $P(0 \le X_2 \le 4) \approx 0.6827$. On the other hand, the actual probabilities are (as determined by software) $P(180 \le X_{200} \le 220) = 0.6835$ and $P(0 \le X_2 \le 4) = 0.8647$.

2a. The distribution of X_n belongs to an exponential family because $f_{X_n}(x)$ has the form $h(x)c(n)\exp[t(x)w(n)]$ with $h(x) := 1_{\{x>0\}}\exp[-x/2]$, $c(n) := (\Gamma[n/2])^{-1}2^{-n/2}$, $t(x) := \log x$, and $w(n) := (n/2-1)$.

2b. Since $\frac{d}{dn}w(n) = 1/2$, we have

$$
\begin{aligned}
(1/2)E[\log X_n] &= E\left[\frac{d}{dn}w(n)t(X_n)\right] \\
&= -\frac{d}{dn}\log c(n) \\
&= -\frac{d}{dn}\{-\log(\Gamma[n/2]) - (n/2)\log 2\} \\
&= (1/2)\psi[n/2] + (1/2)\log 2,
\end{aligned}
$$

so that $E[\log X_n] = \psi[n/2] + \log 2$.

2c. Since $\frac{d}{dn}w(n) = 1/2$ and $\frac{d^2}{dn^2}w(n) = 0$, we have

$$
\begin{aligned}
(1/4)Var[\log X_n] &= Var\left[\frac{d}{dn}w(n)t(X_n)\right] \\
&= -\frac{d^2}{dn^2}\log c(n) \\
&= -\frac{d^2}{dn^2}\{-\log(\Gamma[n/2]) - (n/2)\log 2\} \\
&= \frac{d}{dn}\{(1/2)\psi[n/2] + (1/2)\log 2\}, \\
&= (1/4)\psi_1[n/2],
\end{aligned}
$$

so that $Var[\log X_n] = \psi_1[n/2]$.

2d. Recalling that $c(n) := (\Gamma[n/2])^{-1}2^{-n/2}$, we have

$$
\begin{aligned}
\int_0^\infty \log[x]x^{n/2-1}\exp[-x/2]\ dx &= c(n)^{-1}\int_0^\infty c(n)\log[x]x^{n/2-1}\exp[-x/2]\ dx \\
&= c(n)^{-1}\int_0^\infty \log[x]f_{X_n}(x)\ dx \\
&= c(n)^{-1}E[\log X_n] \\
&= \Gamma[n/2]2^{n/2}\{\psi[n/2] + \log 2\}.
\end{aligned}
$$

2e. Putting $u := 2\lambda x$ and $du := 2\lambda\ dx$, and identifying α with $n/2$, we have

$$
\begin{aligned}
\int_0^\infty \log[x]x^{\alpha-1}\exp[-\lambda x]\ dx &= \int_0^\infty \log[u/(2\lambda)][u/(2\lambda)]^{\alpha-1}\exp[-u/2]\ [du/(2\lambda)] \\
&= (2\lambda)^{-\alpha}\int_0^\infty \{\log[u] - \log[2\lambda]\}u^{\alpha-1}\exp[-u/2]\ du \\
&= (2\lambda)^{-\alpha}\left\{\int_0^\infty \log[u]u^{\alpha-1}\exp[-u/2]\ du - \log[2\lambda]\int_0^\infty u^{\alpha-1}\exp[-u/2]\ du\right\} \\
&= (2\lambda)^{-\alpha}\{\Gamma[\alpha]2^\alpha(\psi[\alpha] + \log 2) - \log[2\lambda]\Gamma[\alpha]2^\alpha\} \\
&= \lambda^{-\alpha}\Gamma[\alpha]\{\psi[\alpha] - \log[\lambda]\}.
\end{aligned}
$$