STA 623 — Fall 2011 — Dr. Charnigo

Written Assignment 4 Solutions

1a. The marginal probability mass function of Y is

$$\begin{aligned} f_Y(y) &= \sum_{x=1}^{\infty} f_{X,Y}(x,y) \\ &= \sum_{x=1}^{\infty} p^2(1-p)^{y-2} 1_{x<y} \\ &= \sum_{x=1}^{y-1} p^2(1-p)^{y-2} \\ &= (y-1)p^2(1-p)^{y-2} \end{aligned}$$

for integers $y \geq 2$. (Since $f_Y(1)$ is clearly 0, the above formula can also be said to apply to integers $y \geq 1$.)

1b. The marginal probability mass function of X is

$$\begin{aligned} f_X(x) &= \sum_{y=1}^{\infty} f_{X,Y}(x,y) \\ &= \sum_{y=1}^{\infty} p^2(1-p)^{y-2} 1_{x<y} \\ &= \sum_{y=x+1}^{\infty} p^2(1-p)^{y-2} \\ &= p(1-p)^{x-1} \end{aligned}$$

for integers $x \geq 1$.

1c. The conditional probability mass function of Y given that $X = x$, a positive integer, is

$$\begin{aligned} f_{Y|X}(y|x) &= f_{X,Y}(x,y)/f_X(x) \\ &= p^2(1-p)^{y-2} 1_{x<y}/\{p(1-p)^{x-1}\} \\ &= p(1-p)^{y-x-1} 1_{x<y} \end{aligned}$$

for integers $y \geq 2$.

1d. Since $f_Y(y)f_X(x) \neq f_{X,Y}(x,y)$ (for example, the left side is $p^3(1-p) > 0$ when $x = y = 2$ while the right side is zero), X and Y are not independent.

1e. Put $Z := Y - X$. For positive integers x and z we have

$$\begin{aligned} P(Z = z|X = x) &= P(Y - X = z|X = x) \\ &= P(Y - x = z|X = x) \\ &= P(Y = z + x|X = x) \\ &= p(1-p)^{z+x-x-1} 1_{x<z+x} \\ &= p(1-p)^{z-1}. \end{aligned}$$

Thus, the conditional probability mass function of Z given that $X = x$, a positive integer, is $f_{Z|X}(z|x) = p(1-p)^{z-1}$ for integers $z \geq 1$.

1f. Since the conditional probability mass function of Z given that $X = x$, a positive integer, does not depend on x, this must also be the marginal probability mass function of Z. (Proof: $f_Z(z) = \sum_{x=1}^{\infty} f_{Z|X}(z|x) f_X(x) = \sum_{x=1}^{\infty} p(1-p)^{z-1} f_X(x) = p(1-p)^{z-1} \sum_{x=1}^{\infty} f_X(x) = p(1-p)^{z-1}$ for integers $z \geq 1$.) Hence, $f_{X,Z}(x,z) = f_{Z|X}(z|x) f_X(x) = f_Z(z) f_X(x)$, so that X and Z are independent.

2a. The marginal probability density function of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx \\ &= \int_{-\infty}^{\infty} 6y 1_{\{0<y<x<1\}}\ dx \\ &= \int_{y}^{1} 6y\ dx = 6y(1-y) \end{aligned}$$

for $y \in (0,1)$.

2b. The marginal probability density function of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy \\ &= \int_{-\infty}^{\infty} 6y 1_{\{0<y<x<1\}}\ dy \\ &= \int_{0}^{x} 6y\ dy = 3x^2 \end{aligned}$$

for $x \in (0,1)$.

2c. The conditional probability density function of Y given that $X = x \in (0,1)$ is

$$\begin{aligned} f_{Y|X}(y|x) &= f_{X,Y}(x,y)/f_X(x) \\ &= 6y 1_{\{0<y<x<1\}}/3x^2 \\ &= 2yx^{-2} 1_{\{0<y<x<1\}}. \end{aligned}$$

2d. Since $f_Y(y) f_X(x) \neq f_{X,Y}(x,y)$ (for example, the left side is $9/8 > 0$ when $x = y = 1/2$ while the right side is zero), X and Y are not independent.