STA 623 — Fall 2013 — Dr. Charnigo

Final Examination

The Final Examination is due on **Tuesday 17 December** at 3 p.m. and is a *strictly individual activity*. This means not only that each person submits his/her own work but also that no discussions about the Final Examination are to take place with anyone other than the instructor.

[60] 1. *Introduction.* Recall that the moment generating function for a random variable X is defined as $M_X(t) := \mathbb{E}[\exp(tX)]$. If one is not intimidated by the imaginary number i (a square root of -1), then one can also define the characteristic function for a random variable X as $C_X(t) := \mathbb{E}[\exp(itX)]$.

In a complex analysis course, one learns the identity that, for real u, $\exp[iu] = \cos[u] + i\sin[u]$. (If you need to persuade yourself that this identity is plausible, then consider what should be the Taylor expansions of the left and right hand sides.) Thus, the characteristic function may be written as $\mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)]$.

Importantly, the characteristic function always exists finitely. This is appealing to statisticians because characteristic functions can be used to give a proof of the Central Limit Theorem requiring fewer assumptions than a proof based on moment generating functions.

In addition, one can attempt to differentiate the characteristic function to acquire various moments of X. However, the results must be adjusted by powers of i. For example, if the second derivative of the characteristic function exists at 0, then this equals -1 times the second moment of X.

In the event that the distribution of X is symmetric about 0, we have $\mathbb{E}[\sin(tX)] = 0$ and the characteristic function simplifies to $\mathbb{E}[\cos(tX)]$. We now consider such a situation by putting $f(x) := (1/2)x^{-2}1_{|x|\geq 1}$, which appeared on your in-class assessment.

[10] a. *Exercise.* For $t \neq 0$, show that

$$\mathbb{E}[\cos(tX)] = |t| \int_{|t|}^{\infty} w^{-2} \cos w \; dw.$$

Hint: First consider $t > 0$ and then appeal to the evenness of the cosine function.

[20] b. *Exercise.* Establish that

$$\lim_{t \to 0^+} \frac{\mathbb{E}[\cos(tX)] - 1}{t} = -\int_0^{\infty} w^{-1} \sin w \; dw$$

and

$$\lim_{t \to 0^-} \frac{\mathbb{E}[\cos(tX)] - 1}{t} = +\int_0^\infty w^{-1} \sin w \; dw.$$

Hint: Integrate by parts, and note that $\cos t - 1 = O(t^2)$ as $t \to 0$.

[10] c. *Exercise.* One can find the exact value of $\int_0^\infty w^{-1} \sin w \; dw$ using the residue calculus from complex analysis. Unfortunately, you do not have the time to learn the residue calculus in the next several days. On the other hand, your STA 605 instructor told me that she taught you about methods of numerical integration ! Therefore, please numerically aproximate $\int_0^\infty w^{-1} \sin w \; dw$ using a suitable method of your choice. Do not submit your computer code, but please provide an English language description of how you implemented the method, in enough detail that I could replicate your work if I were so inclined.

[20] d. *Exercise.* Conclude that

$$\frac{d\mathbb{E}[\cos(tX)]}{dt}\Big|_{t=0}$$

does not exist. Even before completing parts b and c, could you have anticipated that this derivative would not exist ? Please explain.

[20] 2. *Introduction.* At one of this semester's departmental colloquia, the speaker was interested in estimating a ratio of the form θ_1/θ_2, with both θ_1 and θ_2 positive but otherwise unknown parameters. The speaker had available a random quantity $X_1 > 0$ to estimate θ_1 and another random quantity $X_2 > 0$ to estimate θ_2. So, the speaker suggested estimating θ_1/θ_2 by X_1/X_2. (Please note, I am not using the same notation that the speaker used. Indeed, I am simplifying the situation somewhat. However, the essential points are preserved.)

One of our faculty members asked the speaker whether there might be any problem in using a ratio of estimators to make an inference about a ratio of parameters. To see what our faculty member had in mind, suppose that $\mathbb{E}[X_1] = \theta_1$ and $\mathbb{E}[X_2] = \theta_2$. Suppose, moreover, that X_1 and X_2 are independent.

Exercise. Now I want you to use Jensen's Inequality to compare $\mathbb{E}[X_1/X_2]$ to θ_1/θ_2.

[20] 3. *Introduction.* Let X_1 and X_2 be Bernoulli random variables. We do not assume that they are independent or identically distributed, but we do assume that all four of the following probabilities are nonzero. Let $a := \mathbb{P}[X_1 = 1, X_2 = 1]$, $b := \mathbb{P}[X_1 = 1, X_2 = 0]$, $c := \mathbb{P}[X_1 = 0, X_2 = 1]$, and $d := \mathbb{P}[X_1 = 0, X_2 = 0]$.

People often use the "odds ratio" $(ad)/(bc)$ to describe the association between X_1 and X_2. An odds ratio of 1 suggests no association, an odds ratio greater than 1 suggests that X_1 and X_2 are more inclined to agree than disagree, and an odds ratio less than 1 suggests that X_1 and X_2 are more inclined to disagree than agree.

On the other hand, interpeting a correlation between two Bernoulli random variables seems difficult. (Imagine that you repeatedly sample X_1 and X_2, then plot the sampled points in a plane whose horizontal dimension corresponds to X_1 and whose vertical dimension corresponds to X_2. Think about what such a plot would look like.)

Exercise. Despite the difficulty in interpretation, I want you to obtain a formula for the correlation between X_1 and X_2, then show that the correlation is 0 if and only if the odds ratio is 1.

Hint. You can set up a quadratic equation for a, but then the identity $b+c-1 = -(a+d)$ reduces your quadratic equation to a linear equation.