STA 623 — Fall 2013 — Dr. Charnigo

Midterm Examination

The Midterm Examination is due on **Tuesday 22 October** at the end of class and is a *strictly individual activity*. This means not only that each person submits his/her own work but also that no discussions about the Midterm Examination are to take place with anyone other than the instructor.

[60] 1. Consider a race with three horses: A, B, and C.

[15] a. Suppose that the race track offers the following wagering opportunities:

- You may bet (any multiple of) $1.00 on horse A, and if horse A wins you will receive your original wager back plus (the same multiple of) $1.00.
- You may bet (any multiple of) $1.00 on horse B, and if horse B wins you will receive your original wager back plus (the same multiple of) $2.00.
- You may bet (any multiple of) $1.00 on horse C, and if horse C wins you will receive your original wager back plus (the same mulitple of) $3.00.

If one interprets the payouts as “odds”, then, for example, horse C is three times as likely to lose as to win. With this interpretation, calculate the implied “probability” for each horse to win. Assuming a negligibly small probability of a tie or any other anomalous occurrence, comment on the plausibility of the implied probabilities. Why do you suppose that the race track provides payouts that seem to violate the axioms of probability ? (If you want a hint for this latter question, then work through part b.)

[15] b. Identify a positive constant K such that the implied probabilities do become plausible after you multiply them by K. Let the results after multiplication by K be denoted by p_1, p_2, and p_3 respectively. Using p_1, p_2, and p_3, calculate the expected value of the winnings (including the return of your original wager) for each $1.00 wagered on Horse A. Do the same for Horse B and Horse C. Comment on the implications for your betting strategy.

[15] c. Suppose (as may be the case if you are an expert handicapper) that you have good reason to believe that p_1, p_2, and p_3 as defined above are not really accurate estimates of the probabilities of horses A, B, and C winning. Comment on the implications for your betting strategy. For example, suppose that you think p_1 is too low. Does that make you

more or less inclined to bet on Horse A ? (Feel free to provide a numerical example, if that helps you to explain your reasoning.)

[15] d. Suppose that $10,000$ people place a wager on the race, and for simplicity suppose that all wagers are in the same amount of \$10. While your answer to part b suggests that the race track's *expected* income is positive, the race track taking a loss is not an impossible contingency. This will indeed happen if, for example, horse A wins with $9,000$ people having bet on horse A. Therefore, propose a strategy to *guarantee* that the race track's income will be positive and equal to a specified amount (say, \$7,692), if the race track is permitted to finalize the payouts *after* all wagers have been placed but *before* the outcome of the race is known. (Feel free to provide a numerical example, if that helps you to explain your reasoning.)

[40] 2. Let σ be a positive real number and μ_1, μ_2 be any real numbers. Also, let α be a real number between 0 and 1. Put $f(x) := (1-\alpha)(2\pi\sigma^2)^{-1/2}\exp[-(x-\mu_1)^2/(2\sigma^2)] + \alpha(2\pi\sigma^2)^{-1/2}\exp[-(x-\mu_2)^2/(2\sigma^2)]$.

[20] a. Show that $f(x)$ defines a valid probability density function. This probability density function is referred to as a mixture of normal probability density functions or, more simply, a normal mixture. The interpretation is that some fraction $(1-\alpha)$ of the population is governed by a normal distribution with mean μ_1 and variance σ^2 while the other fraction α is governed by a normal distribution with mean μ_2 and variance σ^2.

[20] b. Let X have the aforementioned probability density function. Calculate $E[X]$ and $V[X]$. In particular, show that $V[X] > \sigma^2$ unless $\alpha(1-\alpha)(\mu_2-\mu_1) = 0$, in which case $f(x)$ simplifies to an ordinary normal probability density function.