STA 623 – Fall 2013 – Dr. Charnigo

Section 3.5: Location and Scale Families

Location family. Let $f(x)$ be any probability density function, and let μ be any real number. Then $f(x;\mu) := f(x-\mu)$ is also a probability density function because, upon substitution of $y := x - \mu$ and $dy := dx$,

$$\int_{-\infty}^{\infty} f(x-\mu)\ dx = \int_{-\infty}^{\infty} f(y)\ dy = 1.$$

We refer to $\{f(x;\mu) : \mu \in \mathbb{R}\}$ as a location family.

Examples (location family). [Normal location family] Put

$$f(x) := (2\pi)^{-1/2} \exp[-x^2/2],$$

and let μ be any real number. Then

$$f(x;\mu) := f(x-\mu) = (2\pi)^{-1/2} \exp[-(x-\mu)^2/2].$$

Geometrically, $f(x;\mu)$ is a rigid horizontal shift of $f(x)$ by the amount μ. In fact, if $X \sim f(x)$, then $Y := X + \mu \sim f(y;\mu)$ because with $g(x) := x + \mu$ we have $g^{-1}(y) = y - \mu$ and $\frac{d}{dy} g^{-1}(y) = 1$. We also have $E[Y] = E[X] + \mu = \mu$, while $Var[Y] = Var[X] = 1$.

[Exponential location family] Put

$$f(x) := \exp[-x] 1_{x \geq 0},$$

and let μ be any real number. Then

$$f(x;\mu) := f(x-\mu) = \exp[-(x-\mu)] 1_{x \geq \mu}.$$

Again, if $X \sim f(x)$, then $Y := X + \mu \sim f(y;\mu)$. Two questions:

1. Suppose you were told that $Y \sim f(y;\mu)$ but not given the value of μ. If you observed $Y = 2$, what would be your guess (or "estimate") of μ?

2. Is the location family $\{f(x;\mu) : \mu \in \mathbb{R}\}$ an exponential family?

[Cauchy location family] Put

$$f(x) := \pi^{-1}/(1+x^2),$$

and let μ be any real number. Then

$$f(x;\mu) := f(x-\mu) = \pi^{-1}/(1+(x-\mu)^2).$$

Once more, if $X \sim f(x)$, then $Y := X + \mu \sim f(y;\mu)$. Two questions:

1. Do we have $E[Y] = E[X] + \mu$?

2. Define the $100p^{th}$ percentile for the distribution of X to be the number x_p such that $P(X \leq x_p) = p$. What are the 25th, 50th, and 75th percentiles for the distribution of X? If we analogously define percentiles for the distribution of Y, how are they related to those for the distribution of X?

Scale family. Let $f(x)$ be any probability density function, and let σ be any positive real number. Then, as you should verify, $f(x;\sigma) := f(x/\sigma)/\sigma$ is also a probability density function. We refer to $\{f(x;\sigma) : \sigma \in (0,\infty)\}$ as a scale family.

Examples (scale family). [Normal scale family] Put

$$f(x) := (2\pi)^{-1/2}\exp[-x^2/2],$$

and let σ be any positive real number. Then

$$f(x;\sigma) := f(x/\sigma)/\sigma = (2\pi)^{-1/2}\sigma^{-1}\exp[-x^2/(2\sigma^2)].$$

Geometrically, $f(x;\sigma)$ is a compression of $f(x)$ if $\sigma < 1$ and a dilation of $f(x)$ if $\sigma > 1$. If $X \sim f(x)$, then, as you should verify, $Y := \sigma X \sim f(y;\sigma)$. We have $E[Y] = E[\sigma X] = \sigma E[X] = 0$ and $Var[Y] = Var[\sigma X] = \sigma^2 Var[X] = \sigma^2$.

[Exponential scale family] Put

$$f(x) := \exp[-x]1_{x\geq 0},$$

and let σ be any positive real number. Then

$$f(x;\sigma) := f(x/\sigma)/\sigma = \sigma^{-1}\exp[-x/\sigma]1_{x\geq 0}.$$

Again, if $X \sim f(x)$, then $Y := \sigma X \sim f(y;\sigma)$.

[Cauchy scale family] Put

$$f(x) := \pi^{-1}/(1+x^2),$$

and let σ be any positive real number. Then

$$f(x;\sigma) := f(x/\sigma)/\sigma = (\pi\sigma)^{-1}/(1+(x/\sigma)^2).$$

Once more, if $X \sim f(x)$, then $Y := \sigma X \sim f(y;\sigma)$. How do the 25th, 50th, and 75th percentiles for the distribution of X relate to those for the distribution of Y?

Location-scale family. Let $f(x)$ be any probability density function. Let μ be any real number, and let σ be any positive real number. Then, as you should verify, $f(x;\mu,\sigma) := f((x-\mu)/\sigma)/\sigma$ is also a probability density function. We refer to $\{f(x;\mu,\sigma) : \mu \in \mathbb{R}, \sigma \in (0,\infty)\}$ as a location-scale family. See Written Assignment 4 from Fall 2009 for an example.

Further remarks. Not all two-parameter families of distributions are location-scale families. (Consider the family of gamma distributions.)

The concept of location-scale families is not particularly useful for discrete random variables because, for the most part, the support sets for different distributions within the same location-scale family would have no elements in common. Consequently, the location-scale family would not coincide with any recognizable parametric family. (Consider building a location-scale family starting from a Poisson probability mass function. If $\mu = 2.6$ and $\sigma = 1.5$, then the support set is $\{2.6, 4.1, 5.6, 7.1, \ldots\}$, which does not contain any integers.)