

# STA 623 – Fall 2013 – Dr. Charnigo

## Section 4.7: Inequalities

*Hölder's Inequality and Cauchy-Schwarz Inequality.* Let  $p$  and  $q$  be positive numbers with  $1/p + 1/q = 1$ . Then, for any random variables  $X$  and  $Y$ , we have

$$|E[XY]| \leq E[|XY|] \leq (E[|X|^p])^{1/p}(E[|Y|^q])^{1/q}$$

whenever all of these expectations exist as finite numbers. Your textbook authors provide the proof.

The special case when  $p = q = 2$  is called the Cauchy-Schwarz Inequality,

$$|E[XY]| \leq E[|XY|] \leq (E[X^2])^{1/2}(E[Y^2])^{1/2}.$$

*Applications of Hölder's Inequality and Cauchy-Schwarz Inequality.*

1. Suppose that  $X$  and  $Y$  are random variables with finite second moments. Can we derive the relationship  $|\rho_{XY}| \leq 1$  from the Cauchy-Schwarz Inequality? We need to exercise some care here since  $E[XY]$  is not the same as  $Cov[X, Y]$  unless  $E[X]$  or  $E[Y]$  happens to equal 0.

2. By letting  $Y$  equal 1 with probability 1, we see that Hölder's Inequality yields

$$E[|X|] \leq (E[|X|^p])^{1/p}$$

at any  $p \in (1, \infty)$  for which both expectations exist as finite numbers. What is the interpretation of this result when  $p = 2$ ?

3. Let  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  be arbitrary real constants. I claim that

$$\sum_{i=1}^n |a_i b_i| \leq \left( \sum_{i=1}^n |a_i|^p \right)^{1/p} \left( \sum_{i=1}^n |b_i|^q \right)^{1/q}.$$

Can you explain how this follows from Hölder's Inequality?

4. A special case of item 3 arises when  $b_1 = \dots = b_n = 1$  and  $p = q = 2$ ,

$$\sum_{i=1}^n |a_i| \leq n^{1/2} \left( \sum_{i=1}^n a_i^2 \right)^{1/2}.$$

Thus, if you are told that  $\sum_{i=1}^n a_{i,n}^2 \rightarrow 0$  as  $n \rightarrow \infty$ , you may conclude that  $n^{-1/2} \sum_{i=1}^n |a_{i,n}| \rightarrow 0$ . (Here I write  $a_{i,n}$  rather than  $a_i$  since the summands may change with  $n$ .) Is this conclusion valid without the absolute value,  $n^{-1/2} \sum_{i=1}^n a_{i,n} \rightarrow 0$ ? Is this conclusion valid without the  $n^{-1/2}$ ,  $\sum_{i=1}^n |a_{i,n}| \rightarrow 0$ ?

*Jensen's Inequality.* Let  $X$  be a random variable whose support set is contained in an interval  $I \subset \mathbb{R}$ , and suppose that  $g$  is a convex function from  $I$  to  $\mathbb{R}$ . That is, suppose that  $g(tu + (1-t)v) \leq tg(u) + (1-t)g(v)$  for  $t \in (0, 1)$ ,  $u \in I$ , and  $v \in I$ . (If  $g$  is twice differentiable, then nonnegativity of the second derivative implies convexity of  $g$ .) Provided that both expectations exist as finite numbers, we have

$$E[g(X)] \geq g(E[X]).$$

Your textbook authors provide the proof.

A function  $g$  is said to be concave if  $-g$  is convex. (If  $g$  is twice differentiable, then nonpositivity of the second derivative implies concavity of  $g$ .) Jensen's

Inequality is reversed for concave  $g$ ,

$$E[g(X)] \leq g(E[X]).$$

*Remarks on Jensen's Inequality.*

1. To remember the direction of the inequality, think of the convex function  $g(x) := x^2$ . You know that  $E[X^2]$  must be greater than or equal to  $(E[X])^2$  because their difference is  $Var[X]$ , which must be nonnegative.

2. In some instances, we may wish to obtain the stronger conclusion that

$$E[g(X)] > g(E[X]).$$

However, the stronger conclusion is not universally valid. For instance, if  $X$  is degenerate or  $g$  is linear, then we have equality. A sufficient condition for the stronger conclusion is that  $X$  not be degenerate and  $g$  have strictly positive second derivative.

*An application of Jensen's Inequality.* Let  $X$  have the gamma distribution with known shape parameter  $n \in \{2, 3, \dots\}$  and unknown rate parameter  $\lambda \in (0, \infty)$ . Suppose that we wish to “guess”  $\lambda$  after observing a realization of  $X$ . Noting that the expected value of  $X$  is  $n/\lambda$ , we may think that the realization of  $X$  is a good guess for  $n/\lambda$  and hence the realization of  $n/X$  is a good guess for  $\lambda$ . While this strategy does not seem unreasonable, on average we will [choose one and then justify: underestimate / overestimate]  $\lambda$ .

Note that we did not need to compute  $E[n/X]$  to conclude that it differed from  $\lambda$ . But, just for fun, let us compute  $E[n/X]$  anyway: