STA 623 — Fall 2013 — Dr. Charnigo

Written Assignment 1 Solutions

1. *Claim.* $\cap_{j=1}^{\infty} \cup_{n=j}^{\infty} \cap_{i=n}^{\infty} A_i = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$, the set of elements appearing in all but finitely many of $A_1, A_2, \ldots$.

Proof. Fix a positive integer $j > 1$. Let $D_j := \cup_{n=j}^{\infty} E_n$ and $E_n := \cap_{i=n}^{\infty} A_i$. If $x \in D_j$, then there exists N $(\geq j \geq 1)$ such that $x \in E_N$. Hence $x \in \cup_{n=1}^{\infty} E_n = D_1$, so that $D_j \subset D_1$.

If $x \in D_1$, then there exists N (≥ 1) such that $x \in E_N$. Consider two cases. If $N \geq j$, then $x \in \cup_{n=j}^{\infty} E_n = D_j$ and so $D_1 \subset D_j$. If $N < j$, then $x \in (\cap_{i=N}^{j-1} A_i) \cap (\cap_{i=j}^{\infty} A_i) \subset \cap_{i=j}^{\infty} A_i = E_j$. Hence $x \in \cup_{n=j}^{\infty} E_n = D_j$ and so $D_1 \subset D_j$.

Since $D_j = D_1$ for any $j > 1$, we have $\cap_{j=1}^{\infty} D_j = \cap_{j=1}^{\infty} D_1 = D_1 = \cup_{n=1}^{\infty} \cap_{i=n}^{\infty} A_i$.

2. *Proof.* Put $C_0 := (\cup_{i=1}^{\infty} C_i)^c = \cap_{i=1}^{\infty} C_i^c$. Then, since $\cup_{i=0}^{\infty} C_i = C_0 \cup (\cup_{i=1}^{\infty} C_i) = (\cup_{i=1}^{\infty} C_i)^c \cup (\cup_{i=1}^{\infty} C_i) = S$ and $C_0 \cap C_j \subset C_j^c \cap C_j = \emptyset$ (hence, $C_0 \cap C_j = \emptyset$) for any positive integer j, we see that $C_0, C_1, C_2, \ldots$ constitute a partition. Result 11 therefore yields $P(A) = \sum_{i=0}^{\infty} P(A \cap C_i) = P(A \cap C_0) + \sum_{i=1}^{\infty} P(A \cap C_i) \geq \sum_{i=1}^{\infty} P(A \cap C_i)$.

Examples. Flip a fair coin twice, so that $S = \{HH, HT, TH, TT\}$ with each element of S assigned 1/4 probability. Put $C_1 := \{HH\}$, $C_2 := \{HT\}$, $C_3 := \{TH\}$, and $C_j := \emptyset$ for $j \geq 4$. Hence, in the notation above, $C_0 := \{TT\}$. Then equality holds if $A \cap C_0 = \emptyset$ (example, $A := \{HH\}$) but does not hold if $A \cap C_0 = \{TT\}$ (example, $A := \{TT\}$).

3. Let A denote the event that you receive a flush and B denote the event that I receive a flush. For simplicity, let us assume that you are dealt all five of your cards before I am dealt any of mine. Then there are $\binom{52}{5}$ hands available to you, of which $4\binom{13}{5}$ yield a flush. (The 4 is needed because the flush could be in hearts, diamonds, clubs, or spades.) Hence $P(A) = 4\binom{13}{5}/\binom{52}{5}$. After you receive your hand, there are $\binom{47}{5}$ hands available to me. If you have been dealt a flush, then there are 8 cards remaining in one suit and 13 cards remaining in each of the other three suits. Thus, among the hands available to me, $\binom{8}{5} + 3\binom{13}{5}$ yield a flush. Hence $P(B|A) = (\binom{8}{5} + 3\binom{13}{5})/\binom{47}{5}$. Finally, the probability that both of us receive a flush is $P(A \cap B) = P(A)P(B|A) = 4\binom{13}{5}(\binom{8}{5} + 3\binom{13}{5})/(\binom{47}{5}\binom{52}{5})$.

4. The requirement $f(x) \geq 0$ necessitates $C_1 \geq 1$. The requirement $\int_{\mathbb{R}} f(x)\ dx = 1$ necessitates $\int_{C_1}^{C_2} \log x\ dx = C_2(\log C_2 - 1) - C_1(\log C_1 - 1) = 1$. The latter equality has infinitely many solutions, for example $(C_1, C_2) = (1, e)$. There is not a simple algebraic formula by which C_2 can be expressed in terms of C_1, so most of these solutions must be found numerically. As you should illustrate using R of other software of your choice, a graph of C_1 and C_2 values defining a valid probability density function asymptotically approaches (from above) the 45 degree line in a C_1C_2-plane. To get some insight into why this is so, note that $\log x$ increases very slowly for large x, so that $\int_{C_1}^{C_2} \log x\ dx \approx \int_{C_1}^{C_2} \log C_1\ dx = (C_2 - C_1)\log C_1$ for large C_1, whence $C_2 \approx C_1 + 1/\log C_1$ for large C_1.

5. The claim is false. Here are two counterexamples.

Counterexample #1. Put $f(x) := (1/2)^{|x|}$ for negative integers x. Then $f(x)$ defines a valid probability mass function since $f(x) \geq 0$ and $\sum_{j=1}^{\infty}(1/2)^j = 1$. Also, $\{x \in \mathbb{R} : f(x) > 0\}$, being the set of negative integers, is infinite. Yet, $F(1) = P(X \leq 1) = 1$.

Counterexample #2. Put $f(x) := x$ for $x \in \{(1/2), (1/4), (1/8), \ldots\}$. Then $f(x)$ defines a valid probability mass function since $f(x) \geq 0$ and $\sum_{j=1}^{\infty}(1/2)^j = 1$. Also, $\{x \in \mathbb{R} : f(x) > 0\}$, being in one-to-one correspondence with the set of positive integers, is infinite. Yet, $F(1) = P(X \leq 1) = 1$.