STA 623 — Fall 2013 — Dr. Charnigo

Written Assignment 2 Solutions

1a. The statement is true. By the definition of limit, there exists $M > 1$ such that $x > M$ implies $q/2 \leq x^{-p}/f_X(x) \leq 3q/2$ and hence $2x^{-p}/(3q) \leq f_X(x) \leq 2x^{-p}/q$. We have

$$\int_0^\infty x^{p-2} f_X(x)\ dx = \int_0^M x^{p-2} f_X(x)\ dx + \int_M^\infty x^{p-2} f_X(x)\ dx, \tag{1}$$

with the first integral on the right side of (1) bounded above by $\int_0^M x^{p-2} r\ dx = M^{p-1} r/(p-1)$ and the second integral bounded above by $\int_M^\infty 2x^{-2}/q\ dx = 2/(qM)$. Thus,

$$0 \leq \int_0^\infty x^{p-2} f_X(x)\ dx \leq M^{p-1} r/(p-1) + 2/(qM),$$

so that $E[X^{p-2}]$ exists finitely.

1b. The statement is true. Let M be as in part a. We have

$$\int_0^\infty x^{p-2} f_X(x)\ dx = \int_0^1 x^{p-2} f_X(x)\ dx + \int_1^M x^{p-2} f_X(x)\ dx + \int_M^\infty x^{p-2} f_X(x)\ dx. \tag{2}$$

Since $x^{p-2} \leq 1$ when $0 \leq x \leq 1$, the first integral on the right side of (2) is bounded above by $\int_0^1 f_X(x)\ dx \leq \int_0^\infty f_X(x)\ dx = 1$. Since $f_X(x)$ is continuous on $(0, \infty)$, $f_X(x)$ is also continuous on the compact interval $[1, M]$. Any continuous function is bounded on a compact interval, and so there exists $t > 0$ such that $f_X(x) \leq t$ on $[1, M]$, even if $f_X(x)$ is unbounded on $(0, \infty)$. Therefore the second integral on the right side of (2) is bounded above by $\int_1^M x^{p-2} t\ dx \leq \int_0^M x^{p-2} t\ dx = M^{p-1} t/(p-1)$. And, as in part a, the third integral is bounded above by $2/(qM)$. Thus,

$$0 \leq \int_0^\infty x^{p-2} f_X(x)\ dx \leq 1 + M^{p-1} t/(p-1) + 2/(qM),$$

so that $E[X^{p-2}]$ exists finitely.

1c. The statement is false. We will show this by demonstrating that $E[X^p] = \infty$, so that $M_X(t)$ cannot exist finitely in a neighborhood of 0 and, hence, the derivative in question does not exist. Let M be as in part a. Since $2/(3q) \leq x^p f_X(x) \leq 2/q$ for $x > M$, we have

$$\int_0^\infty x^p f_X(x)\ dx \geq \int_M^\infty x^p f_X(x)\ dx \geq \int_M^\infty 2/(3q) = \infty.$$

2a. We have $E[X] = \int_0^\infty \lambda^2 x^2 \exp[-\lambda x]\ dx$. Put $u := \lambda x^2$ and $dv := \lambda \exp[-\lambda x]\ dx$ for integration by parts, yielding $E[X] = -\lambda x^2 \exp[-\lambda x]|_0^\infty + \int_0^\infty 2\lambda x \exp[-\lambda x]\ dx$. The first piece is zero (apply L'Hopital's Rule if that is unclear), and the second piece can be addressed with another integration by parts. We obtain $E[X] = -2x \exp[-\lambda x]|_0^\infty + \int_0^\infty 2 \exp[-\lambda x]\ dx$. The first piece is zero, and the second piece is readily seen to be $2/\lambda$.

2b. We recognize $x^2 \exp[-\lambda x]$ as the kernel of the gamma probability density function with shape parameter 3 and rate parameter λ. The normalizing constant for this kernel is $\lambda^3/2$, so that $E[X] = \int_0^\infty \lambda^2 x^2 \exp[-\lambda x]\ dx = 2/\lambda \int_0^\infty (\lambda^3/2) x^2 \exp[-\lambda x]\ dx = 2/\lambda$.

2c. We have $E[X] = \int_0^\infty \exp[-\lambda x](1+\lambda x)\ dx = \int_0^\infty \exp[-\lambda x]\ dx + \int_0^\infty \lambda x \exp[-\lambda x]\ dx$. The first piece is readily seen to be $1/\lambda$. The second piece can be integrated by parts (as in the second step of part a), handled using the kernel method (noting that the normalizing constant for the gamma probability density function with shape parameter 2 and rate parameter λ is λ^2), or recognized as the integral defining the mean of an exponential random variable with rate λ. In any case, one obtains $1/\lambda$, whence $E[X] = 2/\lambda$.

2d. The moment generating function is $M_X(t) = 1/(1-t/\lambda)^2$ for $t < \lambda$. We obtain $M_X'(t) = (2/\lambda)/(1-t/\lambda)^3$ and hence $E[X] = M_X'(0) = 2/\lambda$.

2e. For real y we have $P(Y \le y) = P(\log X \le y) = P(X \le \exp[y]) = 1 - \exp[-\lambda \exp(y)](1 + \lambda \exp[y])$.

2f. For real y we have $\frac{d}{dy} P(Y \le y) = f_Y(y) = \lambda^2 \exp(2y) \exp[-\lambda \exp(y)]$.

2g. For real y we have $f_Y(y) = f_X(\exp[y]) \frac{d}{dy} \exp[y] = \lambda^2 \exp(2y) \exp[-\lambda \exp(y)]$.