STA 623 — Fall 2013 — Dr. Charnigo

Written Assignment 3 Solutions

1a. [This is not the only possible solution.] We have $f(x;1,1,1,1,1) = 1_{0<x<1} = f(x;0,1,1,1,1)$ even though $(1,1,1,1,1)^T \neq (0,1,1,1,1)^T$.

1b. We have $f(x;0.5,2,1,1,2) = 0.5 \times 2x1_{0<x<1} + 0.5 \times 2(1-x)1_{0<x<1} = 1_{0<x<1}$ with $\gamma(1-\gamma)(\alpha_1 - \alpha_2)(\beta_1 - \beta_2) = 0.5 \times 0.5 \times 1 \times -1 = -0.25 \neq 0$.

1c. [This is not the only possible solution.] Suppose that

$$f(x;\gamma,\alpha_1,\alpha_2,1,1) = f(x;\gamma',\alpha_1',\alpha_2',1,1) \tag{1}$$

with both $\gamma(1-\gamma)(\alpha_1-\alpha_2) > 0$ and $\gamma'(1-\gamma')(\alpha_1'-\alpha_2') > 0$. Using (1) we have

$$\lim_{x\to 0^+} x^{1-\alpha_2} f(x;\gamma',\alpha_1',\alpha_2',1,1) = \lim_{x\to 0^+} x^{1-\alpha_2} f(x;\gamma,\alpha_1,\alpha_2,1,1) = \gamma\alpha_2 \in (0,\infty). \tag{2}$$

On the other hand, $\lim_{x\to 0^+} x^{1-\alpha_2} f(x;\gamma',\alpha_1',\alpha_2',1,1)$ would be 0 if $\alpha_2' > \alpha_2$ and would be ∞ if $\alpha_2' < \alpha_2$, which forces $\alpha_2' = \alpha_2$. Thus, we have

$$\lim_{x\to 0^+} x^{1-\alpha_2} f(x;\gamma',\alpha_1',\alpha_2',1,1) = \lim_{x\to 0^+} x^{1-\alpha_2'} f(x;\gamma',\alpha_1',\alpha_2',1,1) = \alpha_2'\gamma' = \alpha_2\gamma'$$

and, in view of (2), $\alpha_2\gamma' = \alpha_2\gamma$, whence $\gamma' = \gamma$. Using (1) and the preceding conclusions that $\alpha_2' = \alpha_2$ and $\gamma' = \gamma$, we have

$$\int_0^1 xf(x;\gamma,\alpha_1,\alpha_2,1,1)\,dx = \int_0^1 xf(x;\gamma',\alpha_1',\alpha_2',1,1)\,dx = \int_0^1 xf(x;\gamma,\alpha_1',\alpha_2,1,1)\,dx. \tag{3}$$

The left member of (3) is $(1-\gamma)\alpha_1/(1+\alpha_1) + \gamma\alpha_2/(1+\alpha_2)$, while the right is $(1-\gamma)\alpha_1'/(1+\alpha_1') + \gamma\alpha_2/(1+\alpha_2)$. The equality in (3) implies $\alpha_1/(1+\alpha_1) = \alpha_1'/(1+\alpha_1')$, whence $\alpha_1 = \alpha_1'$. In summary, our supposition in (1) led to the conclusion that $(\gamma,\alpha_1,\alpha_2,1,1) = (\gamma',\alpha_1',\alpha_2',1,1)$, which is to say that θ is identifiable under the indicated restrictions.

2a. Putting $u := \tau x^p$ and $du := p\tau x^{p-1}\,dx$, we have $\int_0^\infty \exp[-\tau x^p]\,dx = \int_0^\infty (p\tau)^{-1}x^{1-p}\exp[-u]\,du = \int_0^\infty (p\tau)^{-1}u^{1/p-1}\tau^{1-1/p}\exp[-u]\,du = p^{-1}\tau^{-1/p}\Gamma[1/p]$. Thus, the normalizing constant for the probability density function of X is $p\tau^{1/p}/\Gamma[1/p]$.

2b. The family indexed by p and τ is not an exponential family; the quantity inside the exponential, namely $-\tau x^p$, cannot be expressed as a product of a function of x with a function of τ and p.

If there were a location parameter, call it $\mu(\tau,p)$, then there would have to exist τ' and p' such that $\mu(\tau',p') = \mu(1,1) - 1$. Since the support set corresponding to $\mu(1,1)$ is $(0,\infty)$, the support set corresponding to $\mu(\tau',p')$ would have to be $(-1,\infty)$. However, no member of the family indexed by p and τ yields the support set of $(-1,\infty)$. Thus, there is no location parameter, and we cannot have a location-scale family.

2c. For fixed p, the family indexed by τ is an exponential family because the probability density function has the representation $h(x)c(\tau)\exp[t(x)w(\tau)]$ with $h(x) := 1_{x>0}$, $c(\tau) := p\tau^{1/p}/\Gamma[1/p]$, $t(x) := x^p$, and $w(\tau) := -\tau$.

This family is a scale family because the probability density function has the representation $g(x/\tau^{-1/p})/\tau^{-1/p}$ with $g(u) := p\exp[-u^p]/\Gamma[1/p]1_{u>0}$.

2d. Suppose that $p > 1$. We must show that $\int_0^\infty \exp[tx]\exp[-\tau x^p]\,dx < \infty$ for any fixed t. (The normalizing constant of the probability density function is irrelevant to whether the integral is finite, so we just ignore it here.) If $t \leq 0$, then from part a we have $\int_0^\infty \exp[tx]\exp[-\tau x^p]\,dx \leq \int_0^\infty \exp[-\tau x^p]\,dx = p^{-1}\tau^{-1/p}\Gamma[1/p] < \infty$. If $t > 0$, then there exists $x^* > 0$ such that $tx - \tau x^p < -(\tau/2)x^p$ for $x > x^*$. We have $\int_0^\infty \exp[tx]\exp[-\tau x^p]\,dx = \int_0^{x^*} \exp[tx]\exp[-\tau x^p]\,dx + \int_{x^*}^\infty \exp[tx]\exp[-\tau x^p]\,dx$. The integral from 0 to x^* is finite since the integrand is bounded on that interval, while the integral from x^* to ∞ is dominated by $\int_{x^*}^\infty \exp[-(\tau/2)x^p]\,dx \leq \int_0^\infty \exp[-(\tau/2)x^p]\,dx = p^{-1}(\tau/2)^{-1/p}\Gamma[1/p] < \infty$.

2e. Suppose that $p < 1$. We must show that $\int_0^\infty \exp[tx]\exp[-\tau x^p]\,dx$ diverges for any $t > 0$. To see this, note that there exists $x^* > 0$ (not necessarily the same as in part d) such that $tx - \tau x^p > (t/2)x$ for $x > x^*$. We have $\int_0^\infty \exp[tx]\exp[-\tau x^p]\,dx \geq \int_{x^*}^\infty \exp[tx]\exp[-\tau x^p]\,dx \geq \int_{x^*}^\infty \exp[(t/2)x]\,dx$, and the latter integral is clearly divergent.

On the other hand, all of the moments exist. To see this, fix k and consider $\int_0^\infty x^k \exp[-\tau x^p]\,dx$. (Again, the normalizing constant is irrelevant to whether the integral is finite, so we just ignore it here.) Putting $u := \tau x^p$ and $du := p\tau x^{p-1}\,dx$, we have $\int_0^\infty x^k \exp[-\tau x^p]\,dx =$
$\int_0^\infty \tau^{-(k+1)/p}p^{-1}u^{(k+1)/p-1}\exp[-u]\,du = \tau^{-(k+1)/p}p^{-1}\Gamma[(k+1)/p] < \infty$.

2f. If $p = 1$, then the family indexed by τ is the familiar exponential scale family.