STA 623 — Fall 2013 — Dr. Charnigo

Written Assignment 4 Solutions

1a. We have $f_X(x)f_Y(y) = 16x(1-x^2)y^3$ for $x \in (0,1)$ and $y \in (0,1)$, which is clearly not the same as $f_{X,Y}(x,y)$. Therefore, X and Y are not independent.

1b. There is no contradiction, since $8xy$ is not the joint probability density function of X and Y. The joint probability density function is $8xy1_{0<x<y<1}$, which is not the product of a function of x with a function of y.

2. For $t \in (-\lambda, \lambda)$ we have $M_X(t) = \int_0^\infty \lambda \exp[-(\lambda - t)x]\ dx = \lambda/(\lambda - t)$.

3. For $t \in (-\lambda, \lambda)$ we have $M_W(t) = \int_{-\infty}^\infty (\lambda/2)\exp[-\lambda|w| + tw]\ dw = \int_{-\infty}^0 (\lambda/2)\exp[(\lambda + t)w]\ dw + \int_0^\infty (\lambda/2)\exp[-(\lambda - t)w]\ dw = (\lambda/2)\{1/(\lambda + t) + 1/(\lambda - t)\} = \lambda^2/(\lambda^2 - t^2)$.

4a. For $t \in (-\lambda, \lambda)$ we have, using exercise 2, $M_{X-Y}(t) = M_X(t) \times M_Y(-t) = \lambda/(\lambda - t) \times \lambda/(\lambda + t) = \lambda^2/(\lambda^2 - t^2)$. This matches $M_W(t)$ as computed in exercise 3, so that $X - Y$ has the same distribution as W. (This distribution is called "double exponential" or "Laplace".)

4b. Put $U := X - Y$ and $V := X + Y$. The support set for the joint probability density function of U and V consists of $(u,v)' \in \mathbb{R}^2$ such that $u + v > 0$ and $v - u > 0$, which is to say that $v > |u| > 0$. We have $X = h_1(U,V) := (U + V)/2$ and $Y = h_2(U,V) := (V - U)/2$, so the matrix of partial derivatives of $h_1(u,v)$ and $h_2(u,v)$ has determinant 1/2 and $f_{U,V}(u,v) = (1/2)f_{X,Y}(h_1(u,v), h_2(u,v)) = (1/2)f_X(h_1(u,v))f_Y(h_2(u,v)) = (\lambda^2/2)\exp[-\lambda v]$ for $v > |u| > 0$. The marginal probability density function of U is then obtained by integration in dv, $f_U(u) = \int_{|u|}^\infty (\lambda^2/2)\exp[-\lambda v]\ dv = (\lambda/2)\exp[-\lambda|u|]$ for $|u| > 0$. We may define $f_U(0)$ arbitrarily, and a convenient choice entails extension by continuity: $f_U(0) = (\lambda/2)$. Finally, we recognize that U has the same distribution as W from exercise 3.

5a. Put $U := X$ and $V := X + Y$. The support set for the joint probability density function of U and V consists of $(u,v)' \in \mathbb{R}^2$ such that $u > 0$ and $v - u > 0$, which is to say that $v > u > 0$. We have $X = h_1(U,V) := U$ and $Y = h_2(U,V) := V - U$, so the matrix of partial derivatives of $h_1(u,v)$ and $h_2(u,v)$ has determinant 1 and $f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) = f_X(h_1(u,v))f_Y(h_2(u,v)) = \lambda^2\exp[-\lambda v]$ for $v > u > 0$.

5b. The marginal probability density function of V is then obtained by integration in du, $f_V(v) = \int_0^v \lambda^2\exp[-\lambda v]\ du = \lambda^2 v\exp[-\lambda v]$ for $v > 0$. Thus, V has the gamma distribution with shape 2 and rate λ.

5c. The conditional probability density function of U given that $V = c$, where c is some positive constant, is calculated as $f_{U,V}(u,c)/f_V(c) = \{\lambda^2\exp[-\lambda c]\}/\{\lambda^2 c\exp[-\lambda c]\} = 1/c$ for $0 < u < c$. Thus, conditional on $V = c$, U has the uniform distribution on $(0,c)$.